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New photon-added and photon-depleted coherent states associated with inverse q-boson operators: nonclassical properties

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Abstract

In this paper, we introduce a new family of photon-added as well as photon-depleted q-deformed coherent states related to the inverse q-boson operators. These states are constructed via the generalized inverse q-boson operator actions on a newly introduced family of q-deformed coherent states (Quesne C 2002 *J. Phys. A: Math. Gen.* **35** 9213) which are defined by slightly modifying the maths-type q-deformed coherent states. The quantum statistical properties of these photon-added and photon-depleted states, such as quadrature squeezing and photon-counting statistics, are discussed analytically and numerically in the context of both conventional (nondeformed) and deformed quantum optics.

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1. Introduction

The conventional coherent states of the quantum harmonic oscillator [1], as well as generalized coherent states associated with various algebras [2], have found considerable applications in quantum optics. The basic operators of the quantum harmonic oscillator are the boson annihilation and creation operators $\hat{a}$ and $\hat{a}^{+}$ satisfying the Heisenberg–Weyl algebra $H(4)$:$[\hat{a}, \hat{a}^{+}] = 1$, $[\hat{N}, \hat{a}^{+}] = \hat{a}^{+}$, $[\hat{N}, \hat{a}] = -\hat{a}$, with $\hat{N} = \hat{a}^{+}\hat{a}$ as the number operator. The conventional coherent states defined as the eigenstates of the operator $\hat{a}$ have properties similar to those of a classical radiation field. The generalized coherent states, in contrast, may exhibit some nonclassical properties, such as squeezing [3] and photon-antibunching [4] or sub-Poissonian statistics [5], which have given rise to ever-increasing interest during the last two decades.

A class of generalized coherent states, which has evoked a lot of interest, is connected with deformations of the canonical commutation relations or, equivalently, with deformed oscillator algebras. By now many kinds of deformed oscillators have been introduced in the literature. All of them can be accommodated within the common mathematical framework of the generalized deformed oscillator [6] which is defined as the algebra generated by the operators $(\hat{\mathrm{I}}, \hat{A}, \hat{A}^{+}, \hat{N})$ satisfying the relations

$$[\hat{A}, \hat{A}^{+}] = \Phi(\hat{N}+1) - \Phi(\hat{N}) \qquad [\hat{N}, \hat{A}^{+}] = \hat{A}^{+} \qquad [\hat{N}, \hat{A}] = -\hat{A} \tag{1}$$

where $\Phi(\hat{N})$ is a positive analytic function with $\Phi(0) = 0$. The structure function $\Phi(x)$ is characteristic of the deformation scheme. Some important examples include the following:

(i) $\Phi(\hat{N}) = \hat{N}$ which gives the usual commutation relation of the Heisenberg–Weyl algebra.

(ii) $\Phi(\hat{N}) = (q^{\hat{N}} - q^{-\hat{N}})/(q - q^{-1})$ which gives the so-called 'physics' q-deformed oscillator first suggested independently by Macfarlane [7], Biedenharn [8], Sun and Fu [9] in connection with the representation theory of quantum group.

(iii) $\Phi(\hat{N}) = (q^{\hat{N}} - 1)/(q - 1)$ which gives the so-called 'maths' q-deformed oscillator first suggested by Arik and Coon [10]. It has since been studied in detail by several authors, e.g. Jannussis *et al* [11] and Kulish and Damaskinsky [12].

Since the introduction of q-deformed coherent states associated with the maths-type and physics-type q-deformed oscillators, several typical q-radiation field states such as q-even (odd) coherent states [13] and q-generalized coherent states [14] have been constructed (for a review see [15]). Few q-deformed coherent states have been endowed with a unity resolution relation. For those connected with maths-type and physics-type q-deformed oscillators, the latter has been written as a q-integral [16] with a weight function expressed in terms of some q-exponential [10, 16]. An alternative formulation has been proposed in terms of an ordinary integral, but the corresponding weight function is then only known through the inverse Fourier transform of some given function [17]. Another type of q-deformed coherent state has also been shown to admit a unity resolution relation in the form of an ordinary integral with a weight function expressed as a Laplace transform of some given function [18].

Recently, a new family of q-deformed coherent states $|z\rangle_q$ has been constructed and studied by Quesne [19]. These states which have been defined by slightly modifying the maths-type q-deformed coherent states, in contrast with known q-deformed coherent states admit a unity resolution relation expressed in terms of an ordinary integral with an explicitly known positive weight function. The structure function of the corresponding deformed oscillator algebra is $\Phi(\hat{N}) \equiv [\hat{N}]_q = (q^{-\hat{N}} - 1)/(1 - q)$. The corresponding deformed annihilation and creation operators, $\hat{b}_q$ and $\hat{b}_q^{+}$, which satisfy the commutation relation $\left[\hat{b}_q, \hat{b}_q^{+}\right] = q^{-\hat{N}-1}$ are related to the nondeformed operators $\hat{a}$ and $\hat{a}^{+}$ through [19]

$$\hat{b}_q = \hat{a}\sqrt{[\hat{N}]_q/\hat{N}} \qquad \hat{b}_q^{+} = \sqrt{[\hat{N}]_q/\hat{N}}\hat{a}^{+}. \tag{2}$$

These operators act on the same Fock-space $\{|n\rangle | n = 0, 1, 2, \ldots\}$ as the nondeformed operators $\hat{a}$ and $\hat{a}^{+}$ such that

$$\hat{b}_q|n\rangle = \sqrt{[n]_q}|n-1\rangle \qquad \hat{b}_q^{+}|n\rangle = \sqrt{[n+1]_q}|n+1\rangle \qquad \hat{b}_q|0\rangle = 0. \tag{3}$$

The state $|z\rangle_q$ is an eigenstate of $\hat{b}_q$, that is $\hat{b}_q|z\rangle_q = z|z\rangle_q$, where $|z\rangle_q$ is expressed as [19]

$$\begin{aligned} |z\rangle_q &= C^{-1/2}(|z|^2, q)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{[n]_q!}}|n\rangle \\ C(|z|^2, q) &= \sum_{n=0}^{\infty}\frac{|z|^{2n}}{[n]_q!} = E_q[(1-q)q|z|^2] \end{aligned} \tag{4}$$

and is labelled by $q \in (0, 1)$. Here z is a complex number, $[n]_q! \equiv [n]_q[n-1]_q \ldots 1$, $[0]_q! \equiv 1$ and $E_q(x) \equiv \prod_{k=0}^{\infty}(1 + q^k x)$ is one of the two q-exponentials introduced by Jackson such that $\lim_{q\to 1^-} E_q[(1-q)z] = \exp(z)$ [21]. It has been shown [19] that such q-deformed coherent states possess some interesting nonclassical properties such as sub-Poissonian photon statistics and quadrature squeezing.

The purpose of the present paper is to construct and discuss the q-analogy of photon-added and photon-depleted states corresponding to the Quesne q-deformed coherent states $|z\rangle_q$. These states are constructed via the generalized inverse q-boson operator actions on the states $|z\rangle_q$. These states, which can mathematically span a complete Hilbert space, reveal nonclassical behaviour different from that of the states $|z\rangle_q$.

The organization of the paper is as follows. In the following section, we first introduce the generalized inverse of the q-creation and q-annihilation operators by their action on the number states, and then discuss their properties via their action on the Quesne q-deformed coherent states $|z\rangle_q$. In sections 3 and 4, respectively, we introduce the photon-added and photon-depleted states corresponding to the Quesne q-deformed coherent states $|z\rangle_q$ as the eigenstates of the combination of q-deformed operators $\hat{b}_q$, $\hat{b}_q^+$ and their generalized inverse operators. The physical properties of these states in the context of conventional quantum optics are studied analytically and numerically in section 5. In section 6 we re-examine the physical properties of those states for q-deformed photons, described by the operators $\hat{b}_q$ and $\hat{b}_q^+$, in the context of deformed quantum optics. Finally, in section 7 we summarize the conclusions.

2. Inverse *q*-boson operators

Following Mehta *et al* [20], one may consider the generalized inverse of q-boson operators which do not, however, possess any inverse in a strict sense because of their singular feature. We define the generalized inverse q-boson operators, $\hat{b}_q^{-1}$ and $\left(\hat{b}_q^+\right)^{-1}$, as follows:

$$\hat{b}_q^{-1}|n\rangle = \frac{1}{\sqrt{[n+1]_q}}|n+1\rangle \tag{5}$$

$$\left(\hat{b}_q^+\right)^{-1}|n\rangle = (1-\delta_{n,0})\frac{1}{\sqrt{[n]_q}}|n-1\rangle. \tag{6}$$

It is easily found that $\hat{b}_q^{-1}$ $\left(\left(\hat{b}_q^+\right)^{-1}\right)$ is the right (left) inverse of $\hat{b}_q$ $\left(\hat{b}_q^+\right)$. Explicitly, it is seen that

$$\hat{b}_q\hat{b}_q^{-1} = \left(\hat{b}_q^+\right)^{-1}\hat{b}_q^+ = \hat{\mathrm{I}} \tag{7}$$

while

$$\hat{b}_q^{-1}\hat{b}_q = \hat{b}_q^+\left(\hat{b}_q^+\right)^{-1} = \hat{\mathrm{I}} - |0\rangle\langle 0| \tag{8}$$

where $|0\rangle\langle 0|$ is the projection operator on vacuum. As is seen, $\hat{b}_q^{-1}$ behaves as a creation operator of a photon, while $\left(\hat{b}_q^+\right)^{-1}$ behaves as an annihilation operator of a photon. Any power of $\hat{b}_q^{-1}$ or $\left(\hat{b}_q^+\right)^{-1}$ is defined by repeated application of the corresponding operator. Particularly, we have

$$\hat{b}_q^m\hat{b}_q^{-m} = \left(\hat{b}_q^+\right)^{-m}\left(\hat{b}_q^+\right)^m = \hat{\mathrm{I}} \tag{9}$$

$$\hat{b}_q^{-m}\hat{b}_q^m = \left(\hat{b}_q^+\right)^m\left(\hat{b}_q^+\right)^{-m} = \hat{\mathrm{I}} - \sum_{j=0}^{m-1}|j\rangle\langle j|. \tag{10}$$

Since the Quesne q-coherent state $|z\rangle_q$ is an eigenstate of $\hat{b}_q$, it would appear that it is also an eigenstate of $\hat{b}_q^{-1}$ with eigenvalue z^{-1}. However, it is not so, as can be seen from

$$\hat{b}_q^{-1}|z\rangle_q = z^{-1}(|z\rangle_q - C^{-1/2}(|z|^2, q)|0\rangle). \tag{11}$$

In fact, being a creation operator, there exists no right eigenstate of the operator $\hat{b}_q^{-1}$. Similarly, $|z\rangle_q$ is not an eigenstate of $\left(\hat{b}_q^{+}\right)^{-1}$, though $\left(\hat{b}_q^{+}\right)^{-1}$ is an annihilation operator,

$$\left(\hat{b}_q^{+}\right)^{-1}|z\rangle_q = zC^{-1/2}(|z|^2, q)\sum_{n=0}^{\infty}\frac{z^n\sqrt{[n]_q!}}{[n+1]_q!}|n\rangle. \tag{12}$$

The only eigenstate of $\left(\hat{b}_q^{+}\right)^{-1}$ is the vacuum state $|0\rangle$ with zero eigenvalue. Therefore the action of the operator $\hat{b}_q^{-1}$ $\left(\left(\hat{b}_q^{+}\right)^{-1}\right)$ on the state $|z\rangle_q$ yields a one-photon excitation (annihilation) state in $|z\rangle_q$. In this manner, the repeated action of $\hat{b}_q^{-1}$ and $\left(\hat{b}_q^{+}\right)^{-1}$ on the state $|z\rangle_q$ yields, respectively, the multiphoton-excitation states

$$\begin{aligned}|z, +m\rangle_q &= C_m^{-1/2}(|z|^2, q)\hat{b}_q^{-m}|z\rangle_q \\ &= \left(C(|z|^2, q) - \sum_{j=0}^{m-1}\frac{|z|^{2j}}{[j]_q!}\right)^{-1/2}\sum_{n=0}^{\infty}\frac{z^{n+m}}{\sqrt{[n+m]_q!}}|n+m\rangle\end{aligned} \tag{13}$$

(with $C_m(|z|^2, q) = 1 - (C(|z|^2, q))^{-1}\sum_{j=0}^{m-1}|z|^{2j}/[j]_q!$) and the multiphoton-annihilation states

$$\begin{aligned}|z, -m\rangle_q &= S_m^{-1/2}(|z|^2, q)\left(\hat{b}_q^{+}\right)^{-m}|z\rangle_q \\ &= \left(\sum_{j=0}^{\infty}\frac{|z|^{2j}[j]_q!}{([j+m]_q!)^2}\right)^{-1/2}\sum_{n=0}^{\infty}\frac{z^n\sqrt{[n]_q!}}{[n+m]_q!}|n\rangle\end{aligned} \tag{14}$$

(with $S_m(|z|^2, q) = (C(|z|^2, q))^{-1}\sum_{j=0}^{\infty}\frac{(|z|^2)^{j+m}}{([j+m]_q!)^2}[j]_q!$) in the state $|z\rangle_q$, where m is a positive integer. It is easy to show that the normalization constants $C_m(|z|^2, q)$ and $S_m(|z|^2, q)$ in (13) and (14) are absolutely convergent series in $|z|^2$ for $0 < q < 1$.

3. *q*-deformed photon-added coherent states

By making use of the q-boson inverse operator $\hat{b}_q^{-1}$ one can construct operators of which the q-deformed photon-added coherent states $|z, +m\rangle_q$ are the right eigenstates. Using $\hat{b}_q^{m}\hat{b}_q^{-m} = \hat{\mathrm{I}}$, (cf equation (9)) and $\hat{b}_q|z\rangle_q = z|z\rangle_q$ one readily finds that the states $|z, +m\rangle_q$ are the eigenstates of $\hat{b}_q^{-m}\hat{b}_q^{m+1}$,

$$\hat{b}_q^{-m}\hat{b}_q^{m+1}|z, +m\rangle_q = z|z, +m\rangle_q. \tag{15}$$

It is clear from (13) that the number states $|j\rangle$ ($j = 0, 1, 2, \ldots, m-1$) are absent in the family of photon-added coherent states $|z, +m\rangle_q$. Therefore, the states $|z, +m\rangle_q$ do not form a complete set. However, each set of them, along with the number states $|j\rangle$ ($j = 0, 1, 2, \ldots, m-1$) does form a complete set, which will be shown later in this section.

On the other hand, there exists another family of q-deformed photon-added coherent states constructed by repeated application of $\hat{b}_q^{+}$ on the states $|z\rangle_q$, i.e.,

$$\begin{aligned}|z, +m\rangle_q' &= (C_m'(|z|^2, q))^{-1/2}\left(\hat{b}_q^{+}\right)^{m}|z\rangle_q \\ &= \frac{C^{1/2}(|z|^2, q)}{\sum_{j=0}^{\infty}\frac{|z|^{2j}[j+m]_q!}{([j]_q!)^2}}\sum_{n=0}^{\infty}\frac{z^n\sqrt{[n+m]_q!}}{[n]_q!}|n+m\rangle\end{aligned} \tag{16}$$

(with $C_m'(|z|^2, q) = C^{-1}(|z|^2, q)\sum_{j=0}^{\infty}|z|^{2j}[j+m]_q!/([j]_q!)^2$). In fact, this family of q-deformed photon-added coherent states can be regarded as an extension of the conventional photon-added coherent states first introduced by Agarwal and Tara [22]. It can be shown that the states $|z, +m\rangle'_q$ are the right eigenstates of $\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m}$, i.e.,

$$\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m} |z, +m\rangle'_q = z|z, +m\rangle'_q. \tag{17}$$

It is clear from expression (17) that the number state $|m\rangle$ (i.e., the state $|0, +m\rangle'_q$) is an eigenstate of $\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m}$ with eigenvalue zero. In fact, all the number states $|0\rangle, |1\rangle, \ldots, |m\rangle$ are the eigenstates of the operator $\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m}$ with the eigenvalue zero. Thus we get $(m+1)$-fold degeneracy for this eigenvalue. Clearly, though the family of states $|z, +m\rangle'_q$ does not form a complete set, we will show that the eigenstates of the operator $\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m}$ do form a complete set. Moreover, It is easy to show that the operator $\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m}$ commutes with the operator $\hat{b}_q - [m]_q q^{-(\hat{N}-m+1)} \left(\hat{b}_q^+\right)^{-1}$,

$$\left[\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m}, \hat{b}_q - [m]_q q^{-(\hat{N}-m+1)} \left(\hat{b}_q^+\right)^{-1}\right] = 0. \tag{18}$$

We conclude from this relation that the operators $\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m}$ and $\hat{b}_q - [m]_q q^{-(\hat{N}-m+1)} \left(\hat{b}_q^+\right)^{-1}$ may have simultaneous eigenstates. In fact, all the non-degenerate eigenstates of $\left(\hat{b}_q^+\right)^m \hat{b}_q \left(\hat{b}_q^+\right)^{-m}$ are also the eigenstates of $\hat{b}_q - [m]_q q^{-(\hat{N}-m+1)} \left(\hat{b}_q^+\right)^{-1}$ with the same eigenvalue z,

$$\left(\hat{b}_q - [m]_q q^{-(\hat{N}-m+1)} \left(\hat{b}_q^+\right)^{-1}\right)|z, +m\rangle'_q = z|z, +m\rangle'_q. \tag{19}$$

In addition, the states $|0\rangle$ and $|m\rangle$ are the two-fold degenerate eigenstates of the operator $\hat{b}_q - [m]_q q^{-(\hat{N}-m+1)} \left(\hat{b}_q^+\right)^{-1}$ with zero eigenvalue. Since the projection of the eigenstates of $\hat{b}_q - [m]_q q^{-(\hat{N}-m+1)} \left(\hat{b}_q^+\right)^{-1}$ on the number states $|1\rangle, |2\rangle, \ldots, |m-1\rangle$ is zero, this family of states, namely the eigenstates of the operator $\hat{b}_q - [m]_q q^{-(\hat{N}-m+1)} \left(\hat{b}_q^+\right)^{-1}$, also does not form a complete set.

By using the completeness relation (resolution of unity relation) of the states $|z\rangle_q$ [19],

$$\int_C \int \mathrm{d}^2 z |z\rangle_q W(|z|^2, q)_q \langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{\mathrm{I}} \tag{20}$$

where

$$W(|z|^2, q) = \frac{1-q}{\pi \ln q^{-1}} \frac{E_q[(1-q)q|z|^2]}{E_q[(1-q)|z|^2]} \tag{21}$$

we find the corresponding completeness relations of the family of the states $|z, +m\rangle_q$. Multiplying (20) by $\hat{b}_q^{-m}$ on the left-hand side and $\hat{b}_q^m$ on the right-hand side we obtain

$$\int_C \int \mathrm{d}^2 z\, \hat{b}_q^{-m} |z\rangle_{qq}\langle z| \hat{b}_q^m W(|z|^2, q) = \hat{\mathrm{I}} - \sum_{i=0}^{m-1} |i\rangle\langle i|. \tag{22}$$

We again multiply equation (22) by $\left(\hat{b}_q^+\right)^{-m} \left(\hat{b}_q^+\right)^m = \hat{\mathrm{I}}$ (cf equation (9)) and use equation (13) to obtain

$$\int_C \int \mathrm{d}^2 z\, C_m(|z|^2, q) |z, +m\rangle_{qq}\langle z, +m| \left(\hat{b}_q^+\right)^m \hat{b}_q^m W(|z|^2, q) + \sum_{i=0}^{m-1} |i\rangle\langle i| = \hat{\mathrm{I}}. \tag{23}$$

This equation shows that the states $|z, +m\rangle_q$, along with the number states $\{|j\rangle, j = 0, 1, 2, \ldots, m-1\}$, span a complete Hilbert space. On the other hand, if we first multiply equation (20) by $\left(\hat{b}_q^+\right)^m$ on the left-hand side and $\hat{b}_q^m$ on the right-hand side and use equation (16) we obtain

$$\int_C\int \mathrm{d}^2z\, C'_m(|z|^2,q)|z,+m\rangle'_q{}'_q\langle z,+m|W(|z|^2,q) = \left(\hat{b}_q^+\right)^m\hat{b}_q^m \tag{24}$$

and then multiplying by $\left(\left(\hat{b}_q^+\right)^m\hat{b}_q^m\right)^{-1}$ we arrive at

$$\int_C\int \mathrm{d}^2z\, C'_m(|z|^2,q)|z,+m\rangle'_q{}'_q\langle z,+m|\left(\left(\hat{b}_q^+\right)^m\hat{b}_q^m\right)^{-1}W(|z|^2,q) + \sum_{j=0}^{m-1}|j\rangle\langle j| = \hat{\mathrm{I}}. \tag{25}$$

This equation, too, shows the completeness of the eigenstates of the operator $\left(\hat{b}_q^+\right)^m\hat{b}_q\left(\hat{b}_q^+\right)^{-m}$.

4. *q*-deformed photon-depleted coherent states

It is easy to show that the states $|z,-m\rangle_q$ (cf equation (14)), in which m photons have been depleted from the Quesne q-deformed states $|z\rangle_q$, are the right eigenstates of $\left(\hat{b}_q^+\right)^{-m}\hat{b}_q\left(\hat{b}_q^+\right)^m$ with eigenvalue z,

$$\left(\hat{b}_q^+\right)^{-m}\hat{b}_q\left(\hat{b}_q^+\right)^m|z,-m\rangle_q = z|z,-m\rangle_q. \tag{26}$$

Moreover, it is interesting to note that the operator $\left(\hat{b}_q^+\right)^{-m}\hat{b}_q\left(\hat{b}_q^+\right)^m$ commutes with the operator $\hat{b}_q + [m]_q q^{-(\hat{N}+1)}\left(\hat{b}_q^+\right)^{-1}$. Both the operators have simultaneous eigenstates $|z,-m\rangle_q$ with eigenvalue z,

$$\left(\hat{b}_q + [m]_q q^{-(\hat{N}+1)}\left(\hat{b}_q^+\right)^{-1}\right)|z,-m\rangle_q = z|z,-m\rangle_q. \tag{27}$$

One can verify that, unlike photon-added states $|z,+m\rangle'_q$, neither of these two operators has any degenerate eigenstate. As expected, therefore, the two operators $\left(\hat{b}_q^+\right)^{-m}\hat{b}_q\left(\hat{b}_q^+\right)^m$ and $\hat{b}_q + [m]_q q^{-(\hat{N}+1)}\left(\hat{b}_q^+\right)^{-1}$ are, in fact, identical.

We now examine the completeness of the states $|z,-m\rangle_q$. Multiplying equation (20) by $\left(\hat{b}_q^+\right)^{-m}$ on the left-hand side and $\hat{b}_q^{-m}$ on the right-hand side gives

$$\int_C\int \mathrm{d}^2z\, S_m(|z|^2,q)|z,-m\rangle_{q\,q}\langle z,-m|W(|z|^2,q) = \left(\hat{b}_q^m\left(\hat{b}_q^+\right)^m\right)^{-1}. \tag{28}$$

We multiply equation (28) by $\hat{b}_q^m\left(\hat{b}_q^+\right)^m$ on the right-hand side and use $\left(\hat{b}_q^m\left(\hat{b}_q^+\right)^m\right)^{-1}\left(\hat{b}_q^m\left(\hat{b}_q^+\right)^m\right) = \hat{\mathrm{I}}$ (cf equation (7)). This leads to the completeness relation of the states $|z,-m\rangle_q$, i.e.,

$$\int_C\int \mathrm{d}^2z\, S_m(|z|^2,q)|z,-m\rangle_{q\,q}\langle z,-m|\left(\hat{b}_q^m\left(\hat{b}_q^+\right)^m\right)W(|z|^2,q) = \hat{\mathrm{I}}. \tag{29}$$

5. Physical properties of the states $|z,+m\rangle_q$ and $|z,-m\rangle_q$ in nondeformed quantum optics

In the present section, we shall proceed to study some of the quantum statistical properties of the states $|z,+m\rangle_q$ and $|z,-m\rangle_q$. The photon-added states $|z,+m\rangle'_q$ can be discussed similarly, but we omit them here. To proceed, we first calculate the expectation values of various operators, namely, $\hat{a}$, $\hat{a}^2$, $\hat{a}^+\hat{a}$ and $(\hat{a}^+)^2\hat{a}^2$, in the states $|z,+m\rangle_q$ and $|z,-m\rangle_q$. For the states $|z,+m\rangle_q$, given by equation (13), one readily obtains

$$\langle\hat{a}\rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} z \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{\sqrt{[n+m]_q![n+m+1]_q!}} \sqrt{n+m+1} \tag{30}$$

$$\langle\hat{a}^2\rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} z^2 \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}\sqrt{(n+m+1)(n+m+2)}}{\sqrt{[n+m]_q![n+m+2]_q!}} \tag{31}$$

$$\langle\hat{a}^+\hat{a}\rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}(n+m)}{[n+m]_q!} \tag{32}$$

$$\langle(\hat{a}^+)^2\hat{a}^2\rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}(n+m)(n+m-1)}{[n+m]_q!} \tag{33}$$

and for the states $|z, -m\rangle_q$, given by equation (14), one has

$$\langle\hat{a}\rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} z \sum_{n=0}^{\infty} \frac{|z|^{2n}\sqrt{[n]_q![n+1]_q!(n+1)}}{[n+m]_q![n+m+1]_q!} \tag{34}$$

$$\langle\hat{a}^2\rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} z^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}\sqrt{[n]_q![n+2]_q!(n+1)(n+2)}}{[n+m]_q![n+m+2]_q!} \tag{35}$$

$$\langle\hat{a}^+\hat{a}\rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{|z|^{2n}[n]_q!}{([n+m]_q!)^2} n \tag{36}$$

$$\langle(\hat{a}^+)^2\hat{a}^2\rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{|z|^{2n}[n]_q!}{([n+m]_q!)^2} n(n-1). \tag{37}$$

These expectation values could not be calculated in closed forms as the sums of the respective series are not apparently known. We calculate them numerically for several values of m, q and $|z|$.

Let us consider the Hermitian quadrature operators

$$\hat{X}_1 = \frac{\hat{a} + \hat{a}^+}{2} \qquad \hat{X}_2 = \frac{\hat{a} - \hat{a}^+}{2\mathrm{i}} \tag{38}$$

with commutation relation $[\hat{X}_1, \hat{X}_2] = \mathrm{i}/2$. Their variances $(\Delta\hat{X}_1)^2$ and $(\Delta\hat{X}_2)^2$ in any state $|\psi\rangle$ satisfy the conventional uncertainty relation

$$(\Delta\hat{X}_1)^2(\Delta\hat{X}_2)^2 \geqslant \tfrac{1}{4}|\langle[\hat{X}_1, \hat{X}_2]\rangle|^2 = \tfrac{1}{16} \tag{39}$$

the lower bound being attained by the vacuum state, for which both variances are equal to 1/4. A state is said to be squeezed [23] for the quadrature $\hat{X}_j$ ($j = 1$ or 2) if $(\Delta\hat{X}_j)^2 < \frac{1}{2}|\langle[\hat{X}_1, \hat{X}_2]\rangle| = \frac{1}{4}$. For the states $|z, +m\rangle_q$ and $|z, -m\rangle_q$ the condition for squeezing in quadrature $\hat{X}_j$ ($j = 1$ or 2) can be expressed as, respectively,

$$F_j \equiv 4(\Delta\hat{X}_j)^2_{(z,+m)} - 1 = 4\left(\left\langle\hat{X}_j^2\right\rangle_{(z,+m)} - (\langle\hat{X}_j\rangle_{(z,+m)})^2\right) - 1 < 0 \qquad (j = 1 \text{ or } 2) \tag{40}$$

$$F'_j \equiv 4(\Delta\hat{X}_j)^2_{(z,-m)} - 1 = 4\left(\left\langle\hat{X}_j^2\right\rangle_{(z,-m)} - (\langle\hat{X}_j\rangle_{(z,-m)})^2\right) - 1 < 0 \qquad (j = 1 \text{ or } 2). \tag{41}$$

Therefore by using expressions (30)–(32) we find that squeezing in the quadrature $\hat{X}_1$ occurs for the states $|z, +m\rangle_q$ whenever

$$\cos 2\theta < \xi_m(|z|^2, q) \qquad \xi_m(|z|^2, q) = \frac{V_m^2(|z|^2, q) - U_m(|z|^2, q)}{P_m(|z|^2, q) - V_m^2(|z|^2, q)} \tag{42}$$

and that in the quadrature $\hat{X}_2$ occurs whenever

$$\cos 2\theta > -\xi_m(|z|^2, q) \tag{43}$$

where

$$\begin{aligned} V_m(|z|^2, q) &= (C_m(|z|^2, q))^{-1/2} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{[n+m]_q!} \sqrt{\frac{(n+m+1)}{[n+m+1]_q}} \\ U_m(|z|^2, q) &= \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m-1}}{[n+m]_q!}(n+m) \\ P_m(|z|^2, q) &= \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{[n+m]_q!} \sqrt{\frac{(n+m+1)(n+m+2)}{[n+m+1]_q[n+m+2]_q}} \end{aligned} \tag{44}$$

and the angle θ defined by $z = |z|\,\mathrm{e}^{\mathrm{i}\theta}$ is limited to the interval from zero to π or from $-\pi/2$ to $\pi/2$. On the other hand, by using expressions (34)–(36) we find that the states $|z, -m\rangle_q$ exhibit squeezing in $\hat{X}_1$ quadrature provided

$$\cos 2\theta < \xi'_m(|z|^2, q) \qquad \xi'_m(|z|^2, q) = \frac{V'^2_m(|z|^2, q) - U'_m(|z|^2, q)}{P'_m(|z|^2, q) - V'^2_m(|z|^2, q)} \tag{45}$$

and in $\hat{X}_2$ quadrature provided

$$\cos 2\theta > -\xi'_m(|z|^2, q) \tag{46}$$

where

$$\begin{aligned} V'_m(|z|^2, q) &= (S_m(|z|^2, q))^{-1/2} \sum_{n=0}^{\infty} \frac{|z|^{2n}[n]_q!\sqrt{(n+1)[n+1]_q}}{([n+m]_q!)^2[n+m+1]_q} \\ U'_m(|z|^2, q) &= \sum_{n=0}^{\infty} \frac{(|z|^2)^{n-1}[n]_q!}{([n+m]_q!)^2} n \\ P'_m(|z|^2, q) &= \sum_{n=0}^{\infty} \frac{|z|^{2n}[n]_q!}{([n+m]_q!)^2} \frac{\sqrt{(n+1)(n+2)[n+1]_q[n+2]_q}}{[n+m+1]_q[n+m+2]_q}. \end{aligned} \tag{47}$$

As expected $\xi_0(|z|^2, q) = \xi'_0(|z|^2, q)$.

Numerical analysis shows that whereas for $m = 0$ (corresponding to the Quesne q-deformed states $|z\rangle_q$) the inequality $|\xi_m(|z|^2, q)| < 1$ is always satisfied, for $m \geqslant 1$ this inequality is satisfied for some values of $|z|$. Further, with increasing m and decreasing q ($0 < q < 1$) the range of values of $|z|$ where $|\xi_m(|z|^2, q)| < 1$ becomes smaller. On the other hand, we find that the inequality $|\xi'_m(|z|^2, q)| < 1$ is satisfied for all values of $|z|$, m and q.

In figures 1(*a*) and (*b*) we have shown, respectively, the three-dimensional plot of F_1 (corresponding to squeezing of $\hat{X}_1$ for the states $|z, +m\rangle_q$) and F_2 (corresponding to squeezing of $\hat{X}_2$ for the states $|z, +m\rangle_q$) versus $x = |z|^2$ and θ, for $q = 9/10$, $m = 1$. As is seen, for larger x the maximum squeezing of $\hat{X}_1$ occurs at $\theta = 0$ ($\theta = \pi$) while the maximum squeezing of $\hat{X}_2$ occurs at $\theta = \pi/2$ ($\theta = -\pi/2$). In figures 2(*a*) and (*b*), respectively, we have plotted the variations of F_1 and F_2 with $x = |z|^2$ for different values of m and q and for $\theta = 0$. It is readily observed that with increasing x, F_1 becomes negative (squeezing of $\hat{X}_1$), while F_2 is positive (unsqueezing of $\hat{X}_2$) for all values of x. Furthermore, with increasing m or decreasing q (increasing deformation) squeezing of $\hat{X}_1$ occurs for larger x.

In figures 3(*a*) and (*b*), respectively, the three-dimensional plots of F'_1 (corresponding to squeezing of $\hat{X}_1$ for the states $|z, -m\rangle_q$) and F'_2 (corresponding to squeezing of $\hat{X}_2$ for the states $|z, -m\rangle_q$) versus $x = |z|^2$ and θ, for $q = 9/10$, $m = 1$ are shown. Similar to the case

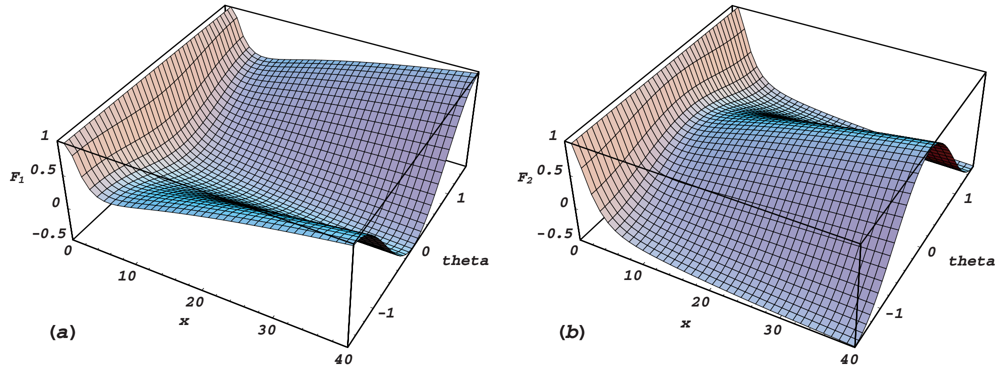

Figure 1. (*a*) Three-dimensional plot of F_1, corresponding to squeezing of $\hat{X}_1$ for the states $|z, +m\rangle_q$, versus $x = |z|^2$ and θ for $q = 9/10$, $m = 1$. (*b*) Three-dimensional plot of F_2, corresponding to squeezing of $\hat{X}_2$ for the states $|z, +m\rangle_q$, versus $x = |z|^2$ and θ for $q = 9/10$, $m = 1$.

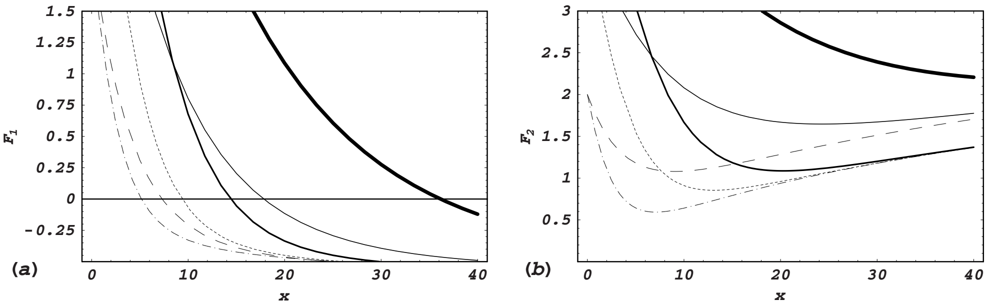

Figure 2. (*a*) Variation of F_1 with $x = |z|^2$ for different values of m and q and for $\theta = 0$; _._._ : $q = 0.8$, $m = 1$; ------ : $q = 0.8$, $m = 2$; ▬▬ : $q = 0.8$, $m = 3$; _ _ _: $q = 0.6$, $m = 1$; _____: $q = 0.6$, $m = 2$; ████ : $q = 0.6$, $m = 3$. (*b*) Variation of F_2 with $x = |z|^2$ for different values of m and q and for $\theta = 0$; _._._ : $q = 0.8$, $m = 1$; ------ : $q = 0.8$, $m = 2$; ▬▬ : $q = 0.8$, $m = 3$; _ _ _ : $q = 0.6$, $m = 1$; _____ : $q = 0.6$, $m = 2$; ████ : $q = 0.6$, $m = 3$.

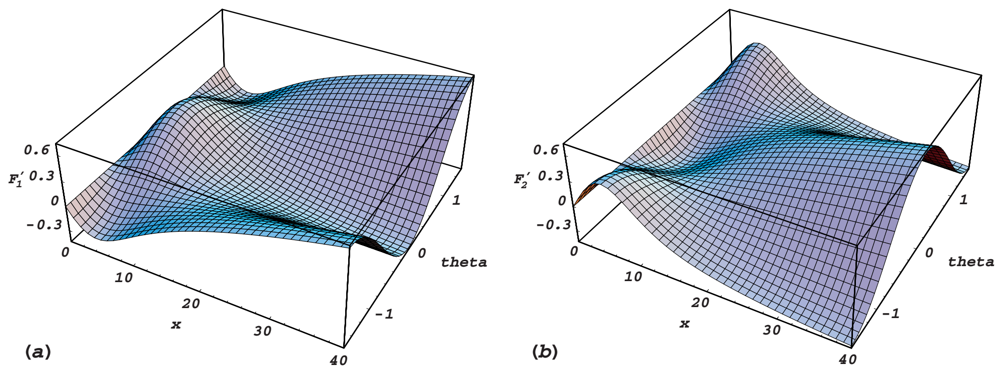

Figure 3. (*a*) Three-dimensional plot of F_1', corresponding to squeezing of $\hat{X}_1$ for the states $|z, -m\rangle_q$, versus $x = |z|^2$ and θ for $q = 9/10$, $m = 1$. (*b*) Three-dimensional plot of F_2', corresponding to squeezing of $\hat{X}_2$ for the states $|z, -m\rangle_q$, versus $x = |z|^2$ and θ for $q = 9/10$, $m = 1$.

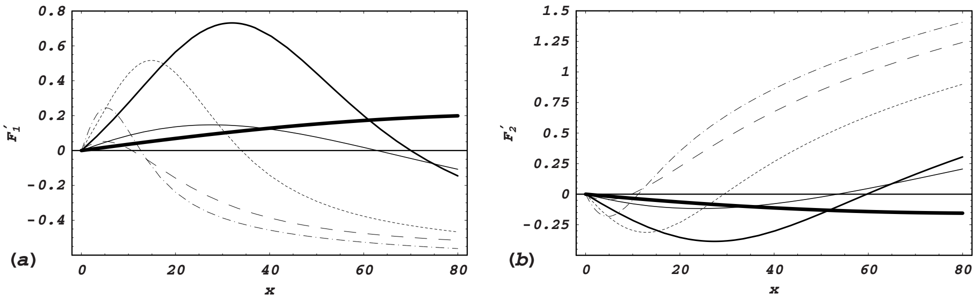

Figure 4. (*a*) Variation of F_1' with $x = |z|^2$ for different values of m and q and for $\theta = 0$; — · — · : $q = 0.8, m = 1$; - - - - - - : $q = 0.8, m = 2$; ▬▬ : $q = 0.8, m = 3$; — — — : $q = 0.6, m = 1$; ——— : $q = 0.6, m = 2$; ▬▬ : $q = 0.6, m = 3$. (*b*) Variation of F_2' with $x = |z|^2$ for different values of m and q and for $\theta = 0$; — · — · : $q = 0.8, m = 1$, - - - - - - : $q = 0.8, m = 2$; ▬▬ : $q = 0.8, m = 3$; — — — : $q = 0.6, m = 1$; ——— : $q = 0.6, m = 2$; ▬▬ : $q = 0.6, m = 3$.

of the states $|z, +m\rangle_q$, for larger x the maximum squeezing of $\hat{X}_1$ occurs at $\theta = 0$ $(\theta = \pi)$ while the maximum squeezing of $\hat{X}_2$ occurs at $\theta = \pi/2$ $(0 = -\pi/2)$. Figures 4(*a*) and (*b*), respectively, show the variations of F_1' and F_2' with $x = |z|^2$ for different values of m and q and for $\theta = 0$. It is observed that for the states $|z, -m\rangle_q$ the quadratures $\hat{X}_1$ or $\hat{X}_2$ may be squeezed such that with increasing x, squeezing of $\hat{X}_1$ is enhanced while squeezing of $\hat{X}_2$ is removed. Moreover, in this case with increasing m or decreasing q (increasing deformation) the range of x where the quadrature $\hat{X}_1$ is squeezed becomes smaller while for $\hat{X}_2$ this range becomes greater.

We now investigate the photon-counting statistics of the states $|z, +m\rangle_q$ and $|z, -m\rangle_q$. For this purpose one needs to determine the Mandel parameter [24],

$$Q = \frac{\langle (\hat{a}^+)^2 \hat{a}^2 \rangle - \langle \hat{a}^+ \hat{a} \rangle^2}{\langle \hat{a}^+ \hat{a} \rangle}. \tag{48}$$

The Mandel parameter $Q < 0$ and $Q > 0$, respectively, corresponds to the terminologies of sub-Poissonian statistics (photon-antibunching effect) and super-Poissonian statistics (photon-bunching effect). For a Poissonian distribution, this parameter is equal to zero. By using expressions (32), (33) and (36), (37) the Mandel parameter reads

$$Q = \frac{\sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}(n+m)(n+m+1)}{[n+m]_q!}}{\sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}(n+m)}{[n+m]_q!}} - (C_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}(n+m)}{[n+m]_q!} \tag{49}$$

for the states $|z, +m\rangle_q$, and

$$Q = \frac{\sum_{n=0}^{\infty} \frac{(|z|^2)^{n+2}[n+2]_q!(n+1)(n+2)}{[n+m+2]_q!}}{\sum_{n=0}^{\infty} \frac{|z|^{2n}[n]_q!n}{([n+m]_q!)^2}} - (S_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{|z|^{2n}[n]_q!n}{([n+m]_q!)^2} \tag{50}$$

for the states $|z, -m\rangle_q$.

We have plotted the variation of the Mandel parameter Q for the states $|z, +m\rangle_q$ and $|z, -m\rangle_q$ versus $x = |z|^2$ in figures 5(*a*) and (*b*), respectively. We conclude from these numerical results that the states $|z, +m\rangle_q$ always show the sub-Poissonian photon-counting statistics, while the states $|z, -m\rangle_q$ always show the super-Poissonian photon-counting statistics. The q-nonlinearity enhances the sub-Poissonian statistics of the states $|z, +m\rangle_q$ for greater x value. In addition, for a fixed value of q the state with greater m value shows enhanced sub-Poissonian behaviour for smaller x value. It should be noted that for $m = 0$, corresponding to the states $|z\rangle_q$, numerical results show that the states are always sub-Poissonian, which is in agreement with the result obtained in [19].

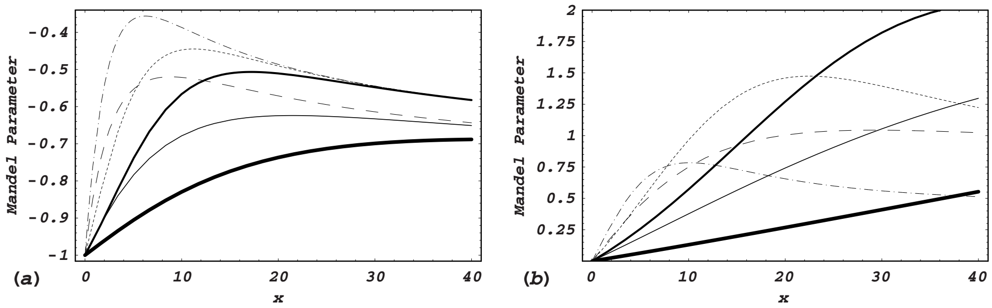

Figure 5. (*a*) Variation of the Mandel parameter Q for the states $|z, +m\rangle_q$ versus $x = |z|^2$; — · — · — : $q = 0.8$, $m = 1$; - - - - - - : $q = 0.8$, $m = 2$; ▬▬ : $q = 0.8$, $m = 3$; — — — : $q = 0.6$, $m = 1$; ——— : $q = 0.6$, $m = 2$; ███ : $q = 0.6$, $m = 3$. (*b*) Variation of the Mandel parameter Q for the states $|z, -m\rangle_q$ versus $x = |z|^2$; — · — · — : $q = 0.8$, $m = 1$; - - - - - - : $q = 0.8$, $m = 2$; ▬▬ : $q = 0.8$, $m = 3$; — — — : $q = 0.6$, $m = 1$; ——— : $q = 0.6$, $m = 2$; ███ : $q = 0.6$, $m = 3$.

6. Physical properties in deformed quantum optics

The q-deformed boson operators $\hat{b}_q$ and $\hat{b}_q^+$ may be interpreted as describing 'dressed' photons, which may be invoked in phenomenological models explaining some observable phenomena [25]. The physical properties considered in section 5 for 'conventional' photons, described by the operators $\hat{a}$ and $\hat{a}^+$, may therefore be re-examined for those deformed photons.

We first investigate the quantum fluctuations of the quadrature operators in the context of deformed quantum optics. For this purpose, we consider the deformed quadrature operators [19]

$$\hat{X}_{1b} = \frac{\hat{b}_q + \hat{b}_q^+}{2} \qquad \hat{X}_{2b} = \frac{\hat{b}_q - \hat{b}_q^+}{2\mathrm{i}}. \tag{51}$$

The two operators $\hat{X}_{1b}$, $\hat{X}_{2b}$ with commutation relation $[\hat{X}_{1b}, \hat{X}_{2b}] = \mathrm{i}q^{-(\hat{N}+1)}/2$ in any state satisfy the uncertainty relation

$$(\Delta\hat{X}_{1b})^2(\Delta\hat{X}_{2b})^2 \geqslant \tfrac{1}{4}|\langle[\hat{X}_{1b}, \hat{X}_{2b}]\rangle|^2 = \langle q^{-2(\hat{N}+1)}\rangle/16. \tag{52}$$

For the states $|z, +m\rangle_q$ and $|z, -m\rangle_q$ the condition for squeezing in quadrature $\hat{X}_{jb}$ ($j = 1$ or 2) can be expressed as, respectively,

$$F_j \equiv 4\left(\left\langle\hat{X}_{jb}^2\right\rangle_{(z,+m)} - (\langle\hat{X}_{jb}\rangle_{(z,+m)})^2\right) - \langle q^{-(\hat{N}+1)}\rangle_{(z,+m)} < 0 \qquad (j = 1 \text{ or } 2) \tag{53}$$

$$F_j' \equiv 4\left(\left\langle\hat{X}_{jb}^2\right\rangle_{(z,-m)} - (\langle\hat{X}_{jb}\rangle_{(z,-m)})^2\right) - \langle q^{-(\hat{N}+1)}\rangle_{(z,-m)} < 0 \qquad (j = 1 \text{ or } 2). \tag{54}$$

It is straightforward to show that for the states $|z, +m\rangle_q$ we have

$$\langle\hat{b}_q\rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} z \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{[n+m]_q!} \tag{55}$$

$$\left\langle\hat{b}_q^2\right\rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} z^2 \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{[n+m]_q!} \tag{56}$$

$$\left\langle\hat{b}_q^+\hat{b}_q\right\rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{[n+m-1]_q!} \tag{57}$$

$$\left\langle \left(\hat{b}_q^+\right)^2 \hat{b}_q^2 \right\rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{[n+m-2]_q!} \tag{58}$$

$$\langle q^{-(\hat{N}+1)} \rangle_{(z,+m)} = (C_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} q^{-(n+m+1)} \frac{(|z|^2)^{n+m}}{[n+m]_q!} \tag{59}$$

and for the states $|z, -m\rangle_q$ we have

$$\langle \hat{b}_q \rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} z \sum_{n=0}^{\infty} \frac{|z|^{2n}[n+1]_q!}{[n+m]_q![n+m+1]_q!} \tag{60}$$

$$\left\langle \hat{b}_q^2 \right\rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} z^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}[n+2]_q!}{[n+m]_q![n+m+2]_q!} \tag{61}$$

$$\left\langle \hat{b}_q^+ \hat{b}_q \right\rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{|z|^{2n}[n]_q![n]_q}{([n+m]_q!)^2} \tag{62}$$

$$\left\langle \left(\hat{b}_q^+\right)^2 \hat{b}_q^2 \right\rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+2}[n+2]_q![n+2]_q[n+1]_q}{([n+m+2]_q!)^2} \tag{63}$$

$$\langle q^{-(\hat{N}+1)} \rangle_{(z,-m)} = (S_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} q^{-(n+1)} \frac{|z|^{2n}[n]_q!}{([n+m]_q!)^2}. \tag{64}$$

Now we easily find that for any $|z|$ and q

$$\begin{aligned} F_1 = F_2 &= (C_m(|z|^2, q))^{-1} \frac{|z|^{2m}}{[m-1]_q!} > 0 \qquad && m \geqslant 1 \\ &= 0 && m = 0 \end{aligned} \tag{65}$$

i.e., there is no squeezing either in $\hat{X}_{1b}$ or $\hat{X}_{2b}$ for the states $|z, +m\rangle_q$. Moreover, we have

$$\begin{aligned} (\Delta \hat{X}_{1b})^2_{(z,+m)} = (\Delta \hat{X}_{2b})^2_{(z,+m)} &= \frac{1}{2}|\langle[\hat{X}_{1b}, \hat{X}_{2b}]\rangle_{(z,+m)}| + \frac{1}{2}(C_m(|z|^2, q))^{-1} \frac{|z|^{2m}}{[m-1]_q!} \qquad && m \geqslant 1 \\ &= \frac{1}{2}|\langle[\hat{X}_{1b}, \hat{X}_{2b}]\rangle_{(z,+m)}| && m = 0 \end{aligned} \tag{66}$$

which indicates that while the states $|z\rangle_q$ are intelligent states for the deformed operators $\hat{X}_{1b}$, $\hat{X}_{2b}$, the photon-added states $|z, +m\rangle_q$ $(m \geqslant 1)$ are not so. On the other hand, we find that the photon-depleted states $|z, -m\rangle_q$ exhibit squeezing in $\hat{X}_{1b}$ quadrature provided

$$\cos 2\theta < \xi'_m(|z|^2, q) \qquad \xi'_m(|z|^2, q) = \frac{W'^2_m(|z|^2, q) - R'_m(|z|^2, q)}{M'_m(|z|^2, q) - W'^2_m(|z|^2, q)} \tag{67}$$

and in $\hat{X}_{2b}$ quadrature provided

$$\cos 2\theta > -\xi'_m(|z|^2, q) \tag{68}$$

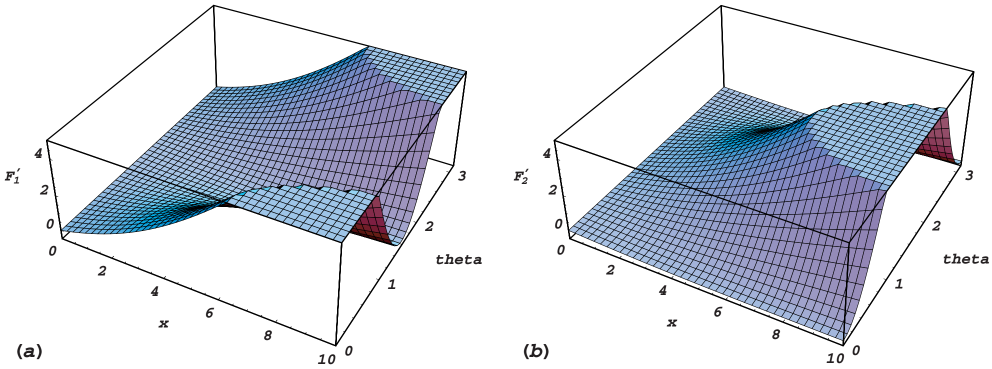

Figure 6. (a) Three-dimensional plot of F_1', corresponding to the squeezing of q-deformed quadrature $\hat{X}_{1b}$ for the states $|z, -m\rangle_q$, versus $x = |z|^2$ and θ for $q = 9/10$, $m = 1$. (b) Three-dimensional plot of F_2', corresponding to the squeezing of q-deformed quadrature $\hat{X}_{2b}$ for the states $|z, -m\rangle_q$, versus $x = |z|^2$ and θ for $q = 9/10$, $m = 1$.

where

$$\begin{aligned} W_m'(|z|^2, q) &= (S_m(|z|^2, q))^{-1/2} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+1}[n+1]_q!}{[n+m]_q![n+m+1]_q!} \\ R_m'(|z|^2, q) &= \sum_{n=0}^{\infty} \frac{|z|^{2n}[n]_q![n]_q}{([n+m]_q!)^2} \\ M_m'(|z|^2, q) &= \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+1}[n+2]_q!}{[n+m]_q![n+m+2]_q!}. \end{aligned} \tag{69}$$

Numerical analysis shows that for $m = 0$ we have $|\xi_m'(|z|^2, q)| > 1$ for any $|z|$ and q, as it is expected, while for $m \geqslant 1$ the inequality $|\xi_m'(|z|^2, q)| < 1$ is always satisfied. In figures 6(a) and (b), respectively, we have depicted the three-dimensional plots of F_1' (corresponding to squeezing of $\hat{X}_{1b}$ for the states $|z, -m\rangle_q$) and F_2' (corresponding to squeezing of $\hat{X}_{2b}$ for the states $|z, -m\rangle_q$) versus $x = |z|^2$ and θ for $q = 9/10$, $m = 1$. It is seen that for $x > 0$ the maximum squeezing of $\hat{X}_{1b}$ occurs at $\theta = \pi/2$ ($\theta = -\pi/2$) while the maximum squeezing of $\hat{X}_{2b}$ occurs at $\theta = 0$ ($\theta = \pi$). In figures 7(a) and (b), respectively, we have plotted the variations of F_1' and F_2' with $x = |z|^2$ for different values of m and q and for $\theta = \pi/2$. From these numerical results, we conclude that for the states $|z, -m\rangle_q$ the deformed quadrature $\hat{X}_{1b}$ is always squeezed while there is no squeezing in $\hat{X}_{2b}$.

Finally, in order to examine photon-counting statistics of deformed photons one can generalize the notion of Mandel parameter and define [19]

$$Q_b = \frac{\left\langle \left(\hat{b}_q^+\right)^2 \hat{b}_q^2 \right\rangle - \left\langle \hat{b}_q^+ \hat{b}_q \right\rangle^2}{\left\langle \hat{b}_q^+ \hat{b}_q \right\rangle} \tag{70}$$

as the q-deformed analogue of the Mandel parameter. By using expressions (57), (58) and (62), (63) it is readily obtained

$$Q_b = \frac{|z|^{2m}}{[m-2]_q!} \left(\sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{[m+n-1]_q!} \right)^{-1} + |z|^2 - (C_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{(|z|^2)^{n+m}}{[m+n-1]_q!} \tag{71}$$

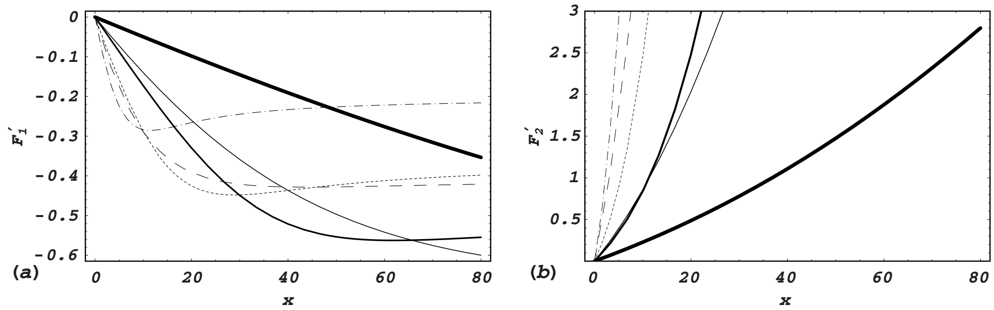

Figure 7. (*a*) Variation of F_1' with $x = |z|^2$ for different values of m and q and for $\theta = \pi/2$; —·—·— : $q = 0.8, m = 1$; ------ : $q = 0.8, m = 2$; ▬▬ : $q = 0.8, m = 3$; — — — : $q = 0.6, m = 1$; ——— : $q = 0.6, m = 2$; ████ : $q = 0.6, m = 3$. (*b*) Variation of F_2' with $x = |z|^2$ for different values of m and q and for $\theta = \pi/2$; —·—·— : $q = 0.8, m = 1$; ------ : $q = 0.8, m = 2$; ▬▬ : $q = 0.8, m = 3$; — — — : $q = 0.6, m = 1$; ——— : $q = 0.6, m = 2$; ████ : $q = 0.6, m = 3$.

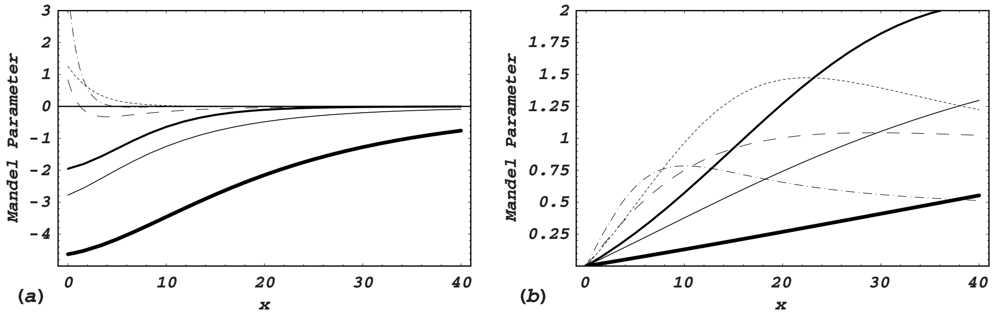

Figure 8. (*a*) Variation of the q-analogy of the Mandel parameter Q_b with $x = |z|^2$ for the states $|z, +m\rangle_q$; —·—·— : $q = 0.8, m = 1$; ------ : $q = 0.8, m = 2$; ▬▬ : $q = 0.8, m = 3$; — — — : $q = 0.6, m = 1$; ——— : $q = 0.6, m = 2$; ████ : $q = 0.6, m = 3$. (*b*) Variation of the q-analogy of the Mandel parameter Q_b with $x = |z|^2$ for the states $|z, -m\rangle_q$; —·—·— : $q = 0.8, m = 1$; ------ : $q = 0.8, m = 2$; ▬▬ : $q = 0.8, m = 3$; — — — : $q = 0.6, m = 1$; ——— : $q = 0.6, m = 2$; ████ : $q = 0.6, m = 3$.

for the states $|z, +m\rangle_q$ and

$$Q_b = \frac{\sum_{n=0}^{\infty} \frac{(|z|^2)^{n+2}[n+2]_q![n+2]_q[n+1]_q}{([n+m+2]_q!)^2}}{\sum_{n=0}^{\infty} \frac{(|z|^2)^n[n]_q![n]_q}{([n+m]_q!)^2}} - (S_m(|z|^2, q))^{-1} \sum_{n=0}^{\infty} \frac{(|z|^2)^n[n]_q![n]_q}{([n+m]_q!)^2} \tag{72}$$

for the states $|z, -m\rangle_q$. We have plotted the variation of the Mandel parameter Q_b with $x = |z|^2$ for the states $|z, +m\rangle_q$ and $|z, -m\rangle_q$ in figures 8(*a*) and (*b*), respectively. We conclude from these numerical results that the states $|z, +m\rangle_q$ may exhibit the sub-Poissonian statistics ($Q_b < 0$), depending on the values of q and m, while the states $|z, -m\rangle_q$ always exhibit the super-Poissonian statistics ($Q_b > 0$). Moreover, the q-nonlinearity enhances the sub-Poissonian behaviour of the states $|z, +m\rangle_q$ for smaller values of $|z|$. It is also to be noted that the numerical results show that for $m = 0$ (corresponding to the states $|z\rangle_q$) we have $Q_b < 0$ for all values of q $(0 < q < 1)$ which is in agreement with the result obtained in [19].

7. Conclusions

In this paper, by using the inverse q-boson annihilation and creation operators $\hat{b}_q^{-1}$, $\left(\hat{b}_q^{+}\right)^{-1}$ associated with a modified version of the maths-type q-deformed oscillator, we have constructed new photon-added and photon-depleted states corresponding to the Quesne q-deformed coherent states $|z\rangle_q$. These states are introduced as the eigenstates of the combination of q-deformed operators $\hat{b}_q$, $\hat{b}_q^{+}$ and their inverse operators. It has been shown that each set of the photon-added states $|z, +m\rangle_q$, along with the number states $\{|j\rangle, j = 0, 1, 2, \ldots, m-1\}$, span a complete Hilbert space. However, each set of the photon-depleted states $|z, -m\rangle_q$ does form a complete set. The nonclassical properties, i.e., sub-Poissonian photon-counting statistics and quadrature squeezing, of the states $|z, +m\rangle_q$ and $|z, -m\rangle_q$ have been studied analytically and numerically in the context of both nondeformed and deformed quantum optics. The results are summarized as follows:

(1) The conventional (nondeformed) quadrature operators $\hat{X}_1$ and $\hat{X}_2$ may be squeezed for both the states $|z, +m\rangle_q$ and $|z, -m\rangle_q$, depending on the values of θ, m and q. In particular, the regions of squeezing are considerably influenced by the q-nonlinearity.

(2) In the context of nondeformed quantum optics, where we consider the conventional photons described by the operators $\hat{a}$ and $\hat{a}^{+}$, the states $|z, +m\rangle_q$ always show the sub-Poissonian photon-counting statistics, while the states $|z, -m\rangle_q$ always show the super-Poissonian photon-counting statistics. The q-nonlinearity enhances the sub-Poissonian statistics of the states $|z, +m\rangle_q$ for greater $x = |z|^2$ values. In addition, for a fixed value of q the state with greater m value shows enhanced sub-Poissonian behaviour for smaller x values.

(3) In the context of deformed quantum optics, where we consider the deformed photons described by the operators $\hat{b}_q$ and $\hat{b}_q^{+}$, the family of the photon-added states $|z, +m\rangle_q$ never shows squeezing either in deformed quadrature operator $\hat{X}_{1b}$ or $\hat{X}_{2b}$. However, the family of the photon-depleted states $|z, -m\rangle_q$ may exhibit squeezing in $\hat{X}_{1b}$ or $\hat{X}_{2b}$.

(4) In the context of deformed quantum optics the states $|z, +m\rangle_q$ exhibit sub-Poissonian statistics ($Q_b < 0$), depending on the values of q and m, while the states $|z, -m\rangle_q$ always exhibit super-Poissonian statistics ($Q_b > 0$). Moreover, the q-nonlinearity enhances the sub-Poissonian behaviour of the states $|z, +m\rangle_q$ for smaller values of $x = |z|^2$.

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